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SUFFICIENT OPTIMALITY CONDITIONS IN DIFFERENTIAL GAMES

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We consider game problems in which the payoff is some function of the terminal state of a conflict-controlled system. We state sufficient conditions for the existence of optimal minimax and maximin strategies of the players. We show that optimal strategies exist if the corresponding Bellman equation has a solution. We consider the question of the existence of optimal strategies both in the class of deterministic as well as in the class of mixed strategies. The reasoning presented is based on the results in [1, 2]. The questions considered border on the investigations presented in [2 - 5].

1. Let the motion of a conflict-controlled system be described by the nonlinear equation

$$dx/dt = f(t, x, u, v) \tag{1.1}$$

Here x is the n-dimensional phase vector, u and v are the controls of the first and second players, respectively, $f(t, x, u, v)$ is a continuous vector-valued function satisfying a Lipschitz condition in x. The realizations $u[t]$ and $v[t]$ of the controls u, and v are constrained by the conditions $u[t] \in P(t)$ and $v[t] \in Q(t)$, where $P(t)$ and $Q(t)$ are closed, bounded and convex sets in the corresponding vector spaces, varying continuously with t. We assume that the right-hand side of system (1.1) satisfies the condition

$$|x'f(t, x, u, v)| \leqslant \lambda(1 + \|x\|^2), \quad u \in P(t), \quad v \in Q(t), \quad t \in [t_0, \vartheta]$$

The payoff is the quantity $w(x[\vartheta])$, defined at the final instant $t = \vartheta$ by the position $x[\vartheta]$ realized. The function $w(x)$ is assumed continuous. Thus, we are considering a game with a fixed final instant $t = \vartheta$. The first player strives to minimize the quantity $w(x[\vartheta])$ under the most adverse behavior of the second player. The second player's problem is to ensure a completion of the game with the largest possible value of the payoff.

We emphasize that the controls u and v should be formed by a feedback rule in order that their realized values $u[t]$ and $v[t]$ be determined at each instant t on the basis of the position $\{t, x[t]\}$ realized at this instant.

In order to include discontinuous control laws and the slipping modes generated by them we introduce the definitions of admissible strategies of the first and second players whose classes we denote $\mathbf{U}_1$ and $\mathbf{V}_1$ respectively. Let a certain set $U(t, x)$ of r-dimensional vectors u be associated with each position $\{t, x\}$. We assume that for any $\{t, x\}$ the set $U(t, x)$ is closed and satisfies the inclusion $U(t, x) \subset P(t)$, while the ambiguous vactor-valued function $U = U(t, x)$ is upper-semicontinuous relative to inclusion with respect to t and x. The latter requirement means that for each position $\{t_*, x_*\}$ and for any number $\alpha > 0$ we can find $\beta > 0$ such that for all x and t satisfying the inequalities $|t - t_*| \leqslant \beta$, $\|x - x_*\| \leqslant \beta$, there holds the inclusion $U(t, x) \subset U_\alpha(t_*, x_*)$, where $U_\alpha(t, x)$ is the α-neighborhood of set $U(t, x)$. We say that the functions $U = U(t, x)$ prescribe the admissible strategies of the first player. The class of functions $V = V(t, x)$ which prescribe the admissible strategies of the second player is defined analogously.

Let us determine the motions of system (1.1), generated by the pair of strategies

$$U \div U(t,x) \in \mathbf{U}_1, \qquad V \div V(t,x) \in \mathbf{V}_1$$

Here the symbol $\div$ denotes correspondence between the strategies $U \in \mathbf{U}_1$ and $V \in \mathbf{V}_1$ and the functions $U = U(t, x)$ and $V = V(t,x)$ prescribing these strategies. By $F(t, x, U, V)$ we denote the convex hull of all vectors of the form $f(t, x, u, v)$, where $u \in U(t, x)$, $v \in V(t, x)$. Every absolutely continuous vector-valued function $x[t]$, $t \geqslant t_0$, which for almost all $t \geqslant t_0$ satisfies the condition

$$dx / dt \in F(t, x[t], U, V), \qquad x[t_0] = x_0$$

is called a motion of system (1.1), $x[t] = x[t; t_0, x_0, U, V]$, generated by the pair of strategies $U \div U(t, x) \in \mathbf{U}_1$, $V \div V(t, x) \in \mathbf{V}_1$. The strategies $U_\tau \div P(t)$, $V_\tau \div Q(t)$ are called the trivial strategies of the first and second players, respectively.

Note that the set of trajectories $x[t] = x[t; t_0, x_0, U_*, V_\tau]$, where U_* is some strategy of the first player, contains any motion $x[t] = x[t; t_0, x_0, U_*, V]$, where V is an arbitrary strategy of the second player. An analogous circumstance obtains for the set of motions $x[t] = x[t; t_0, x_0 U_\tau, V_*]$. The existence of motions $x[t] = x[t; t_0, x_0, U, V]$ prolongable upto the instant $t = \vartheta$ follows from the results in the theory of differential equations with a discontinuous right-hand side [1].

2. We consider the solution of game minimax maximin problems for player strategy classes $\mathbf{U}_1$ and $\mathbf{V}_1$. A strategy satisfying the condition

$$\min_U \max_x w(x[\vartheta; t_0, x_0, U, V_\tau]) = \max_x w(x[\vartheta; t_0, x_0, U^\circ, V_\tau]) \qquad (2.1)$$

is called a minimax strategy of the first player, $U^\circ \div U(t, x) \in \mathbf{U}_1$. Here, in the first case, the maximum is computed over all points $x[\vartheta] = x[\vartheta; t_0, x_0, U, V_\tau]$, while in the second case, over all points $x[\vartheta] = x[\vartheta; t_0, x_0, U^\circ, V_\tau]$. Analogously, the second player's maximin strategy is given by the relation

$$\max_V \min_x w(x[\vartheta; t_0, x_0, U_\tau, V]) = \min_x w(x[\vartheta; t_0, x_0, U_\tau, V^\circ]) \qquad (2.2)$$

The following assertion is valid.

Theorem 1. Let there exist a continuously-differentiable function $\gamma(t, x)$ which for all t and x satisfies the equation

$$\min_u \max_v \left(\frac{\partial\gamma(t,x)}{\partial t} + \mathrm{grad}_x' \gamma(t,x) f(t,x,u,v)\right)_{u,v} = 0 \tag{2.3}$$

$$(u \in P(t),\ v \in Q(t),\ t_0 \leqslant t \leqslant \vartheta)$$

and the boundary condition

$$\gamma(\vartheta,\ x) = w(x[\vartheta]) \tag{2.4}$$

Let $U^\circ(t,\ x)$ be the set of vectors u° resulting in the minimum in (2.3). Then the strategy $U^\circ \div U^\circ(t,\ x) \in \mathbf{U}_1$ is the first player's minimax strategy.

Proof. We show first of all that $U \div U^\circ(t,x)$ belongs to the set $\mathbf{U}_1$. The condition $U^\circ(t,x) \subset P(t)$ follows from the definition of the function $U^\circ = U^\circ(t,\ x)$. Let us verify the fulfillment of the semicontinuity condition for the function $U^\circ = U^\circ(t,\ x)$. We assume the contrary, i.e., let there exist a point $p_* = \{t_*,\ x_*\}$, a sequence $p_k = \{t_k,\ x_k\}$ $(k = 1,\ 2,\ \ldots)$ converging to the point p_*, and a number $\alpha > 0$ such that the sets $U^\circ(t_k,\ x_k)$ are not contained in the set $U_\alpha^\circ(t_*,\ x_*)$ for all $k = 1,\ 2,\ \ldots$. Then there exists the sequence $\{u_k\}$

$$u_k \in U^\circ(t_k,\ x_k), \qquad u_k \notin U_\alpha{}^\circ(t_*, x_*) \qquad (k = 1, 2, \ldots)$$

The sets $U^\circ(t_k,\ x_k)$ are equibounded and, therefore, from the sequence u_k we can pick out a convergent subsequence which we denote, for simplicity, as before by u_k. Let $u_k \to u_*$ as $k \to \infty$. Obviously,

$$u_* \notin U_\alpha{}^\circ(t_*,\ x_*) \tag{2.5}$$

On the other hand, the relation

$$\max_v s(p_k) f(p_k,\ u_k,\ v) \leqslant \max_v s(p_k) f(p_k,\ u,\ v) \tag{2.6}$$

$$v \in Q(t_k), \qquad s(p_k) = (\mathrm{grad}_x'\ \gamma(t, x[t]))_{t_k, x_k}$$

is fulfilled for any element $u \in P(t_k)$. We can show that the function

$$g(u,\ p_k) = \max_v s(p_k) f(p_k,\ u,\ v), \qquad v \in Q(t_k)$$

is continuous relative to $\{u,\ p_k\}$. Therefore, by passing to the limit in (2.6) as $k \to \infty$, we obtain

$$\max_v s(p_*) f(p_*,\ u_*,\ v) \leqslant \max_v s(p_*) f(p_*,\ u,\ v) \qquad v \in Q(t_*)$$

where u is an arbitrary element of set $P(t_*)$. Consequently, $u_* \in U^\circ(t_*,\ x_*)$, which contradicts (2.5). The contradiction proves the semicontinuity of the function $U^\circ = U^\circ(t,\ x)$. The closedness of the sets $U^\circ(t,x)$ is proved in an analogous manner.

Let us now prove that the strategy constructed is minimax. At first we show that for any motion $x[t] = x[t;\ t_0,\ x_0,\ U^\circ,\ V_\tau]$ the condition

$$d\gamma(t,\ x[t]) / dt \leqslant 0 \tag{2.7}$$

is fulfilled for almost all $t \in [t_0,\ \vartheta]$. Here the derivative is computed along the motion $x[t]$. The existence of this derivative for almost all $t \geqslant t_0$ follows from the continuous differentiability of the function $\gamma(t,\ x)$ and the absolute continuity of the motions $x[t]$. The validity of the inequality

$$\frac{\partial\gamma(t,\ x)}{\partial t} + \mathrm{grad}_x'\ \gamma(t,\ x[t]) f(t,\ x[t],\ u^\circ,\ v) \leqslant 0 \tag{2.8}$$

for any vectors $u^\circ \in U^\circ(t, x[t])$, $v \in Q(t)$, follows from relation (2.3) and from the definition of the set $U^\circ(t, x)$

Let us show that from (2.8) there follows the relation

$$\frac{\partial \gamma(t, x)}{\partial t} + \mathrm{grad}_x{}' \gamma(t, x[t]) f[t] \leqslant 0 \tag{2.9}$$

for any vector $f[t] \in F(t, x[t], U^\circ, V_\tau)$. By the Carthéodory theorem, any element $f[t]$ from $F(t, x[t], U^\circ, V_\tau)$ can be represented in the form

$$f[t] = \sum_{i=1}^{n+1} \lambda_i f(t, x[t], u_i^\circ, v_i) \tag{2.10}$$

$$u_i^\circ \in U^\circ(t, x[t]), \qquad v_i \in Q; \quad \lambda_i \geqslant 0, \qquad \sum_{i=1}^{n+1} \lambda_i = 1$$

Therefore, the validity of inequality (2.9), from which follows inequality (2.7), yields from relations (2.8) and (2.10). Thus, inequality (2.7) is proven. From this inequality we obtain that the strategy $U^\circ \div U^\circ(t, x)$ guarantees the first player that the game terminates with a payoff satisfying the inequality

$$w(x[\vartheta]) \leqslant \gamma(t_0, x_0) \tag{2.11}$$

Let us show that the result guaranteed the first player by strategy U° is the best one among all those which may be ensured him by a strategy U from the class U_1 being considered. To do this it is sufficient to prove the following: whatever be the strategy $U_* \div U_*(t, x) \in \mathrm{U}_1$ of the first player, among the motions $x[t] = x[t; t_0, x_0, U_*, V_\tau]$ we can find a motion $x_*[t]$ for which the inequality

$$d\gamma(t, x_*[t]) / dt \geqslant 0 \tag{2.12}$$

is fulfilled for almost all t $(t_0 \leqslant t \leqslant \vartheta)$. From this inequality it follows that $\gamma(\vartheta, x_*[\vartheta]) \geqslant \gamma(t_0, x_0)$. Consequently, the strategy U_* cannot guarantee the first player a game termination with a payoff less than $\gamma(t_0, x_0)$.

Let us prove the proposition stated above. We construct the desired motion $x_*[t]$ as a limit transition from Euler polygonal lines $x_k[t]$ $(k = 1, 2, \ldots)$, which we define in the following way. We divide the interval $[t_0, \vartheta]$ into k semi-intervals $[t_0 + i\Delta_k, t_0 + (i+1)\Delta_k)$, where $\Delta_k = (\vartheta - t_0)/k$ $(k = 1, 2, \ldots)$. At the instant $t_i = t_0 + i\Delta_k$ we choose a certain vector $u_i \in U_*(t_i, x_k[t_i])$ and we select a vector v_i so as to satisfy the inequality

$$\frac{\partial \gamma(t_i, x_k[t_i])}{\partial t} + \mathrm{grad}_x{}' \gamma(t_i, x_k[t_i]) f(t_i, x_k[t_i], u_i, v_i) \geqslant 0$$

The possibility of selecting such a vector v_i follows from relation (2.7). The constant controls $u_i = u[t]$, $v_i = v[t]$, where $t \in [t_0 + i\Delta_k, t_0 + (i+1)\Delta_k)$, determine motions of system (1.1) upto the instant $t_{i+1} = t_0 + (i+1)\Delta_k$. At the instant $t = t_{i+1}$ we repeat the above-described procedure for choosing the players' controls. Next, from the sequence of Euler polygonal lines $x_k[t]$ we choose a certain convergent subsequence. The limit of this sequence is denoted by $x_*[t]$. By means of the reasoning used to prove the existence theorems for solutions of differential equations in contingencies [1, 2], we can show that the constructed trajectory $x_*[t]$ is one of the motions $x[t] = x[t; t_0, x_0, U_*, V_\tau]$. Furthermore, from the construction of the polygonal lines $x_k[t]$ it follows that inequality (2.12) holds for the motion constructed. Thus, we have proven that the strategy $U^\circ \div U^\circ(t, x)$ guarantees the first player a game termination with a payoff satisfying the inequality $\gamma(\vartheta, x[\vartheta]) \leqslant \gamma(t_0, x_0)$, and this result is the best one for the strategy class being considered.

The next theorem can be proved in analogous fashion.

Theorem 2. Let there exist a continuously-differentiable function $\gamma^*(t, x)$ which for all t and x satisfies the equation

$$\max_v \min_u \left(\frac{\partial \gamma^*(t, x)}{\partial t} + \mathbf{grad}_x' \gamma^*(t, x) f(t, x, u, v) \right)_{u, v} = 0 \qquad (2.13)$$

$$(u \in P(t), \ v \in Q(t), \ t_0 \leqslant t \leqslant \vartheta)$$

and the boundary condition

$$\gamma^*(\vartheta, x) = w(x[\vartheta]) \qquad (2.14)$$

Let $V^\circ(t, x)$ be the set of vectors v^c resulting in the maximum in (2.13). Then the strategy $V \div V^\circ(t, x)$ is the second player's maximin strategy.

Theorems 1 and 2 give a strict interpretation of the considerations presented briefly in [3] and answer the question posed in [5]. It is not difficult to prove the validity of the following theorem by using the theorems presented above.

Theorem 3. Let there exist a continuously-differentiable function $\gamma(t, x)$ satisfying the equation

$$\min_u \max_v \left(\frac{\partial \gamma(t, x)}{\partial t} + \mathbf{grad}_x' \gamma(t, x) f(t, x, u, v) \right) = \qquad (2.15)$$

$$= \max_v \min_u \left(\frac{\partial \gamma(t, x)}{\partial t} + \mathbf{grad}_x' \gamma(t, x) f(t, x, u, v) \right) = 0$$

$$(u \in P(t), \ v \in Q(t), \ t_0 \leqslant t \leqslant \vartheta)$$

and the boundary condition

$$\gamma(\vartheta, x) = w(x) \qquad (2.16)$$

Let $U^\circ(t, x)$, $V^\circ(t, x)$ be sets of all vectors $u \in P(t)$, $v \in Q(t)$ resulting in the minimax and the maximin in (2.15). Then

$$U^\circ \div U^\circ(t, x) \in \mathbf{U}_1, \ V^\circ \div V^\circ(t, x) \in \mathbf{V}_1,$$

and these strategies are the minimax and the maximin strategies of the first and second players, respectively, and moreover the strategy pair $\{U^\circ \ V^\circ\}$ supplies the saddle point of the game being considered.

3. Sufficient conditions for the existence of the players' optimal strategies in the strategy classes $\mathbf{U}_1$ and $\mathbf{V}_1$ have been derived above. Let us describe other wider strategy classes $\mathbf{U}_2$ and $\mathbf{V}_2$ of the first and second players, respectively. Sufficient existence conditions for optimal strategies will be considered for such classes too. The distinction of the strategies $U \in \mathbf{U}_2$ $(V \in \mathbf{V}_2)$ from the strategies $U \in \mathbf{U}_1$ $(V \in \mathbf{V}_1)$ considered above is that the functions $U = U(t, x)$, $(V = V(t, x))$ associate with the position $\{t, x\}$ of the game not a point set from $P(t)$ $(Q(t))$ but a certain set of probability measures given on $P(t)$ $(Q(t))$. Here the functions $U = U(t, x)$, $V = V(t, x)$ should satisfy the condition of weak upper-semicontinuity relative to inclusion with respect to the variable $p = \{t, x\}$. This condition means the following. Let there be given a sequence of probability measures $\mu_k(du)$ $(\nu_k(dv))$ given on the set $P(t_k)$ $(Q(t_k))$. Let $\mu_k(du) \in U(t_k, x_k)$ $(\nu_k(dv) \in V(t_k, x_k))$ $(k = 1, 2, \ldots)$ and let $\{t_k, x_k\} \to \{t_*, x_*\}$, while the sequence of probability measures $\mu_k(du)$ $(\nu_k(dv))$ converges weakly to the probability measure $\mu_*(du)$ $(\nu_*(dv))$ given on the set $P(t_*)$

$(Q(t_*))$. Then, the inclusion $\mu_*(du) \in U(t_*, x_*)$ $(\nu_*(dv) \in V(t_*, x_*))$ ought to be fulfilled. Recall that a sequence of probability measures $\mu_k(du)$ converges weakly to measure $\mu_*(du)$, if the relation

$$\int g(u)\mu_k(du) \to \int g(u)\mu_*(du)$$

holds for any continuous function $g(u)$.

For the determination of the motions $x[t] = x[t;\ t_0,\ x_0,\ U,\ V]$ generated by strategies $U \in \mathbf{U}_2$, $V \in \mathbf{V}_2$, the set $F(t, x, U, V)$ is defined as the convex hull of the set of all vectors f of the form

$$f = \iint f(t,\ x,\ u,\ v)\,\mu(du)\,\nu(dv)$$

$$(\mu(du) \in U(t,\ x),\ \ \nu(dv) \in V(t,\ x)) \tag{3.1}$$

The trivial strategies U_τ, V_τ are prescribed by the functions $U_\tau = U_\tau(t)$, $V_\tau = V_\tau(t)$, which associate with the variable t the set of all probability measures $\mu(du)$, $\nu(dv)$ given on the sets $P(t)$ $Q(t)$ respectively. The minimax and the maximin strategies $U^\circ \div U^\circ(t,\ x)$, $V^\circ \div V^\circ(t,\ x)$ are given by the conditions (2.1), (2.2), and the sets $U^\circ(t,\ x)$ and $V^\circ(t,\ x)$ are considered from the players' strategy classes $\mathbf{U}_2$ and $\mathbf{V}_2$ respectively. The following assertion is valid.

Theorem 3.1. Let there exist a continuously differentiable function $\gamma = \gamma(t,\ x)$ satisfying the equation

$$\partial\gamma(t,x)/\partial t + \min_\mu \max_\nu \operatorname{grad}_x{}'\gamma(t,x)\iint f(t,x,u,v)\,\mu(du)\,\nu(dv) = \tag{3.2}$$

$$= \partial\gamma(t,x)/\partial t + \max_\nu \min_\mu \operatorname{grad}_x{}'\gamma(t,x)\iint f(t,x,u,v)\,\mu(du)\,\nu(dv)$$

and the boundary condition

$$\gamma(\vartheta,\ x) = w(x) \tag{3.3}$$

Here the minimum and the maximum are computed over the set of probability measures $\mu(du)$ given on $P(t)$ and $\nu(dv)$ given on $Q(t)$, respectively. Let $U^\circ(t,\ x)$, $V^\circ(t,\ x)$ be sets of probability measures $\mu(du)$, $\nu(dv)$ given on $P(t)$, $Q(t)$, which supply the saddle point [6]

$$\min_\mu \max_\nu \iint \operatorname{grad}_x{}'\gamma(t,x)\,f(t,x,u,v)\,\mu(du)\,\nu(dv) = \tag{3.4}$$

$$= \max_\nu \min_\mu \iint \operatorname{grad}_x{}'\gamma(t,x)\,f(t,x,u,v)\,\mu(dv)\,\nu(dv)$$

Then the strategies $U^\circ \div U^\circ(t,\ x) \in \mathbf{U}_2$, $V^\circ \div V^\circ(t,\ x) \in \mathbf{V}_2$ are the minimax and the maximin strategies, respectively. The strategy pair $\{U^\circ,\ V^\circ\}$ supplies the saddle point of the game being considered.

The proof of this theorem is carried out in just the same way as for the preceding Theorems. Note that in the formulation of Theorem 3.1 we have not assumed the existence of saddle point (3.4). From the results in game theory [6] it follows that such a saddle point (3.4) always exists.

4. In conclusion we present a simple example illustrating the theorems derived. Let the motion of a conflict-controlled object be described by the equation

$$dx\,/\,dt = uv, \qquad x(0) = 0$$

where x is a scalar, and the controls u and v may take two values: $+1$ and -1. As the payoff we choose the quantity $\gamma(\vartheta,\ x) = x[\vartheta]$. The game is played on a specified time interval $t \in [0, 1]$.

We first consider the minimax problem in the strategy class U_1. We set up the Bellman equation

$$\min_u \max_v \left(\frac{\partial\gamma(t, x)}{\partial t} + \frac{\partial\gamma(t, x)}{\partial x} uv\right) = 0$$

under the condition $\gamma(1, x) = x$. It is not difficult to compute the minimax on the left-hand side of this equation. We obtain

$$\frac{\partial\gamma(t, x)}{\partial t} + \left|\frac{\partial\gamma(t, x)}{\partial x}\right| = \frac{\partial\gamma(t, x)}{\partial t} + \frac{\partial\gamma(t, x)}{\partial x} = 0$$

$$\gamma(1, x) = x$$

The function $\gamma(t, x) = x - t + 1$ satisfies this equation and the boundary condition. By virtue of Theorem 1 we now obtain that the result

$$\gamma(t_0, x_0) = x_0 - t_0 + 1 = 1$$

is the best one for the first player. Analogously we can show that the best result for the second player is $\gamma(t_0, x_0) = x_0 + t_0 - 1 = -1$.

Let us now consider the minimax and the maximin problems for the strategy classes U_2 and V_2. We show that in the case being considered the function $\gamma(t, x) = x$ satisfies Eq. (3.2) and boundary condition (3.3). We have

$$\min_\mu \max_\nu \frac{\partial\gamma(t, x)}{\partial t} + \frac{\partial\gamma(t, x)}{\partial x} \iint uv\mu(du)\,\nu(dv) =$$

$$= \min_\mu \max_\nu \iint uv\mu(du)\,\nu(dv) = \min_\mu \max_\nu \int u\mu(du) \int v\nu(dv) =$$

$$= \max_\nu \min_\mu \int u\mu(du) \int v\nu(dv) = 0$$

Recall that here $\mu(du)$ and $\nu(dv)$ are probability measures given on a set consisting of the two points $+1$ and -1. The validity of the equality

$$\iint uv\mu(du)\,\nu(dv) = \int u\mu(du) \int v\nu(dv)$$

follows from Fubini's theorem [7]. By $\mu^\circ(du)$, $\nu^\circ(dv)$ we denote the measures supplying the saddle point (3.4). It is not difficult to note that

$$\mu^\circ(u : u = +1) = \mu^\circ(u : u = -1) = 1/2$$

$$\nu^\circ(v : v = +1) = \nu^\circ(v : v = -1) = 1/2$$

By virtue of Theorem 3.1 we obtain that in the example being considered there exists in the strategy classes U_2 and V_2 a value of the game equal to the amount $\gamma(t_0, x_0) = x_0 = 0$. Thus, in this example a value of the game does not exist in the strategy classes U_1 and V_1, and the minimax and maximin payoffs are equal $+1$ and -1 respectively. In the strategy classes U_2, V_2 a value of the game exists which is equal to zero.

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ON INVESTIGATING THE STABILITY OF NEARLY-CRITICAL SYSTEMS

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In [1 - 3] the proof was given of the reduction principle in stability theory when investigating critical cases. In the present paper the reduction principle is proved for nearly-critical cases [4]. The stability problem in one essentially singular case is solved. The stability of a pitch gyro is investigated.

1. We consider a real autonomous system of differential equations of a perturbed motion of the form

$$x_\nu^{\bullet} = \sum_{l=1}^{r} a_{\nu l} x_l + X_\nu(x) \qquad (\nu = 1, \ldots, r, \ x \equiv x_1, \ldots, x_r) \tag{1.1}$$

Here X_ν are holomorphic functions in the region

$$x_1^2 + \ldots + x_r^2 \leqslant H \tag{1.2}$$

whose expansions do not contain terms of less than second order. H is some finite positive number. We assume that the characteristic equation of system (1.1) has q roots with negative real parts, m zero roots, and p roots with real parts which are small in absolute value. We remark that any system with an arbitrary number of zero and pure-imaginary roots and roots with small positive real parts can be reduced to such a form.

Under these conditions system (1.1) can be transformed by means of linear substitutions to the form

$$\begin{aligned} y_s^{\bullet} &= \sum_{k=1}^{n} g_{sk} y_k + Y_s(y, z) \qquad & \delta = k_1 + \ldots + k_n, \quad s = 1, \ldots, n \\ & & j = 1, \ldots, q, \quad n = m + p \\ z_j^{\bullet} &= \sum_{i=1}^{q} p_{ji} z_i + \sum_{\delta \geqslant 2}^{\infty} A_j^{(*)} y^k + Z_j(y, z) \qquad & n + q = r, \quad y^k = y_1^{k_1} \ldots y_n^{k_n} \end{aligned} \tag{1.3}$$